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Journal of Geometry and Physics 56 (2006) 1782–1789
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On the point spectrum of the Dirac operator on a non-compact manifold

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Received 16 September 2005; accepted 12 October 2005
Available online 22 November 2005

Abstract

The absense of the point spectrum of the Dirac operator is investigated. By the method of an integral identity, it is shown that some non-compact complete manifold with a pole has no L^2-eigenspinors for non-zero eigenvalues.

Subject classification codes: Real and complex geometry; Spinors and twistors

MSC: 53C20; 53C21; 53C27

Keywords: Dirac operator; Point spectrum; Radial curvature

1. Introduction

The purpose of this paper is to study the point spectrum of the (classical) Dirac operator on complete non-compact Riemannian manifolds.

The Laplace-Beltrami operator on a complete non-compact Riemannian manifold is an unbounded essentially self-adjoint operator on the space of functions with compact support, and the spectrum of its closure was investigated intensively ([2,3,5,9,11], etc.). To show the absence of the point spectrum in some geometric assumptions, [4,6] used a method originally due to [13]. This method, which relies on an integral identity, was extended to the case of differential forms in [7] too.

Recently the spectrum of the Dirac operator is also investigated by many authors ([1], etc.). In this paper, we apply the method in [4,6,7] to the Dirac operators on complete non-compact manifolds.

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0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.10.006

If a Riemannian manifold M has a spin structure, we can construct the spinor bundle $\mathcal{S}$ on M. It is a Hermitian bundle of left modules over the bundle of Clifford algebras together with a metric connection ∇ which is compatible with Clifford multiplication by tangent vectors ([12], p. 114). The Dirac operator $\mathcal{D} : C^\infty(\mathcal{S}) \to C^\infty(\mathcal{S})$ is defined by the equation

$$\mathcal{D}\sigma = \sum_i e_i \cdot \nabla_{e_i}\sigma$$

where $\{e_i\}$ is an orthonormal bases of the tangent space at a point of M and "·" denotes the Clifford product.

The Dirac operator with domain $C_0^\infty(\mathcal{S})$ is an essentially self-adjoint operator on $L^2(\mathcal{S})$ if M is complete [14]. The domain of its closure, which will be denoted by the same letter is $W^1(\mathcal{S})$. By the elliptic regularity, eigenspinors of $\mathcal{D}$ belong to $C^\infty(\mathcal{S})$.

A point p of a Riemannian manifold M is said to be a pole if and only if the exponential mapping $\exp_p$ is a diffeomorphism from $T_p(M)$ to M [8]. In this paper, we consider only a spin manifold with a pole. It is contractible and hence has a unique spin structure. Our main result shown in Theorem 3. In this theorem, zero eigenvalue is not excluded in general. To determine whether a complete non-compact manifold has harmonic spinors or not seems to demand a different sort of consideration from the case of non-zero eigenvalues.

2. Integral identity

In this section, we show an integral identity for the Dirac operator which is the key ingredient in our argument. This kind of identity was used by many mathematicians in the problems of the Laplacians for functions and differential forms.

Let M be a Riemannian spin manifold of dimension n (not necessarily with a pole). The Hermitian metric $\langle\ ,\ \rangle$ and a metric connection ∇ on its spinor bundle $\mathcal{S}$ satisfy the following properties: for any spinors φ, ψ and any tangent vector v, the identity

$$\langle v \cdot \varphi, \psi\rangle + \langle \varphi, v \cdot \psi\rangle = 0$$

holds. For any vector field X and any spinor field ψ. we have the identity

$$\nabla(X \cdot \psi) = (\nabla X) \cdot \psi + X \cdot \nabla\psi.$$

Let us take a vector field X and a spinor field ψ on M. We define a spinor field $\mathcal{X}(\psi)$ by the following identity:

$$\mathcal{X}(\psi) = \sum_{i,j} \langle e_i, \nabla_{e_j} X\rangle\, e_j \cdot \nabla_{e_i}\psi.$$

It is evident that the right hand side does not depend on the choice of an orthonormal frame $\{e_i\}$.

Lemma 1. *Let X and ψ be a vector field and a spinor field respectively. Then we have the following identity*:

$$\operatorname{Re}\langle \mathcal{D}\psi, \mathcal{D}(\nabla_X\psi)\rangle = \operatorname{Re}\langle \mathcal{D}\psi, \mathcal{X}(\psi)\rangle + \frac{1}{2}\operatorname{div}(|\mathcal{D}\psi|^2 X) - \frac{1}{2}(\operatorname{div} X)|\mathcal{D}\psi|^2 + \frac{1}{2}\operatorname{Re}\langle \mathcal{D}\psi, \operatorname{Ric}(X) \cdot \psi\rangle.$$

Proof. To begin with, we recall the following identity which is called the $\frac{1}{2}$-Ricci formula in Lemma 1.2 of [10]:

$$\frac{1}{2}\mathrm{Ric}(X)\cdot\psi = \mathcal{D}(\nabla_X\psi) - \nabla_X(\mathcal{D}\psi) - \sum_j e_j\cdot\nabla_{(\nabla_{e_j}X)}\psi$$

where Ric denotes the Ricci transformation, i.e.,

$$\mathrm{Ric}(X) = \sum_i R(X, e_i)e_i.$$

From this formula, we have

$$\begin{aligned}\mathrm{Re}\langle\mathcal{D}\psi, \mathcal{D}(\nabla_X\psi)\rangle &= \mathrm{Re}\langle\mathcal{D}\psi, \nabla_X(\mathcal{D}\psi)\rangle + \frac{1}{2}\,\mathrm{Re}\,\langle\mathcal{D}\psi, \mathrm{Ric}(X)\cdot\psi\rangle\\ &\quad + \mathrm{Re}\left\langle\mathcal{D}\psi, \sum_j e_j\cdot\nabla_{(\nabla_{e_j}X)}\psi\right\rangle.\end{aligned}$$

Inasmuch as

$$\begin{aligned}\mathrm{Re}\langle\mathcal{D}\psi, \nabla_X(\mathcal{D}\psi)\rangle &= \frac{1}{2}(\langle\mathcal{D}\psi, \nabla_X(\mathcal{D}\psi)\rangle + \langle\nabla_X(\mathcal{D}\psi), \mathcal{D}\psi\rangle)\\ &= \frac{1}{2}X(|\mathcal{D}\psi|^2) = \frac{1}{2}\mathrm{div}(|\mathcal{D}\psi|^2X) - \frac{1}{2}(\mathrm{div}X)|\mathcal{D}\psi|^2,\\ \sum_j e_j\cdot\nabla_{(\nabla_{e_j}X)}\psi &= \sum_{i,j}\langle e_i, \nabla_{e_j}X\rangle e_j\cdot\nabla_{e_i}\psi = \mathcal{X}(\psi),\end{aligned}$$

the proof is completed. □

Lemma 2. *Let D be a bounded domain with smooth boundary in M. Then for any spinor field ψ, any vector field X and any real number λ, we have*

$$\begin{aligned}&\int_D(|\mathcal{D}\psi|^2 - \lambda^2|\psi|^2)(\mathrm{div}X) - 2\mathrm{Re}\int_D\langle\mathcal{D}\psi, \mathcal{X}(\psi)\rangle + 2\int_D\langle\mathcal{D}^2\psi - \lambda^2\psi, \nabla_X\psi\rangle\\ &\quad - \mathrm{Re}\int_D\langle\mathcal{D}\psi, \mathrm{Ric}(X)\cdot\psi\rangle = \int_{\partial D}(|\mathcal{D}\psi|^2 - \lambda^2|\psi|^2)\langle X, \nu\rangle + 2\mathrm{Re}\int_{\partial D}\langle\nu\cdot\mathcal{D}\psi, \nabla_X\psi\rangle\end{aligned}$$

where ν denotes the outward unit normal to ∂D.

Proof. The divergence theorem implies the following identities:

$$\begin{aligned}&\mathrm{Re}\int_D\langle\mathcal{D}^2\psi, \nabla_X\psi\rangle = \mathrm{Re}\int_D\langle\mathcal{D}\psi, \mathcal{D}(\nabla_X\psi)\rangle + \mathrm{Re}\int_{\partial D}\langle\nu\cdot\mathcal{D}\psi, \nabla_X\psi\rangle,\\ &\mathrm{Re}\int_D\langle\psi, \nabla_X\psi\rangle + \frac{1}{2}\int_D(\mathrm{div}X)|\psi|^2 = \frac{1}{2}\int_D\mathrm{div}(|\psi|^2X) = \frac{1}{2}\int_{\partial D}|\psi|^2\langle X, \nu\rangle.\end{aligned}$$

Consequently, we obtain from Lemma 1 that

$$\begin{aligned}&2\mathrm{Re}\int_D\langle\mathcal{D}^2\psi - \lambda^2\psi, \nabla_X\psi\rangle - \lambda^2\int_D(\mathrm{div}X)|\psi|^2\\ &\quad = 2\mathrm{Re}\int_D\langle\mathcal{D}\psi, \mathcal{D}(\nabla_X\psi)\rangle + 2\mathrm{Re}\int_{\partial D}\langle\nu\cdot\mathcal{D}\psi, \nabla_X\psi\rangle - \lambda^2\int_{\partial D}|\psi|^2\langle X, \nu\rangle\end{aligned}$$

$$= 2\mathrm{Re}\int_D \langle \mathcal{D}\psi, \mathcal{X}(\psi)\rangle + \int_D \mathrm{div}(|\mathcal{D}\psi|^2 X) - \int_D |\mathcal{D}\psi|^2 \mathrm{div} X + \mathrm{Re}\int_D \langle \mathcal{D}\psi, \mathrm{Ric}(X)\cdot\psi\rangle$$

$$-\lambda^2 \int_{\partial D} |\psi|^2 \langle X, \nu\rangle + 2\mathrm{Re}\int_{\partial D} \langle \nu\cdot\mathcal{D}\psi, \nabla_X\psi\rangle.$$

This implies the desired equation immediately. □

It is easily shown that for any tangent vector v and any spinor φ, the term $\langle\varphi, v\cdot\varphi\rangle$ is purely imaginary. Hence, we get the following equation if ψ is an eigenspinor of $\mathcal{D}$ for an eigenvalue λ:

$$\lambda\,\mathrm{Re}\int_D \langle\psi, \mathcal{X}(\psi)\rangle = -\lambda\,\mathrm{Re}\int_{\partial D}\langle\nu\cdot\psi, \nabla_X\psi\rangle.$$

Let M be a complete Riemannian manifold with a pole p_0, and let r denotes the distance function from the point p_0. For every positive number t, we define

$$D(t) = \{x \in M | r(x) < t\}.$$

Proposition 1. *Assume that a vector field X satisfies the inequality*

$$|X| \le C_1 r + C_2$$

for some positive constants C_1 and C_2. If ψ is an L^2 eigenspinor for an eigenvalue λ, then the following equation holds:

$$\lambda\,\mathrm{Re}\int_M \langle\psi, \mathcal{X}(\psi)\rangle = 0.$$

Remark. The assumption $|X| \le C_1 r + C_2$ is satisfied, for example, if the inequality $|\nabla X| \le C$ holds ([4], Lemma 2.4).

Proof. If $|\psi|$ and $|\nabla\psi|$ belong to L^2, then from the co-area formula there exists a sequence $\{r_k\}$ such that

$$\lim_{k\to\infty} r_k \int_{\partial D(r_k)} (|\psi|^2 + |\nabla\psi|^2) = 0$$

([4] p. 146, [6] p. 14). Since the domain of the closure of the Dirac operator consists of L^2 spinor fields with L^2 first derivative, and

$$|\langle\nu\cdot\psi, \nabla_X\psi\rangle| \le \mathrm{const.}\, r_k(|\psi|^2 + |\nabla\psi|^2)$$

on ∂D_k for large k, we obtain that

$$\lim_{k\to\infty}\int_{\partial D(r_k)} \langle\nu\cdot\psi, \nabla_X\psi\rangle = 0$$

for an L^2 eigenspinor ψ. Noting that the function $\langle\psi, \mathcal{X}(\psi)\rangle$ is integrable, we complete the proof. □

Of particular importance is the case $X = \nabla f$ for some function f on M. Let us define for every spinor ψ,

$$\mathcal{H}_f(\psi) = \sum_{i,j} (\mathcal{H}_f)(e_i, e_j) e_j \cdot \nabla_{e_i}\psi$$

where $\mathrm{H}f$ is the Hessian of f and $\{e_i\}$ $(i = 1, 2, \ldots, n)$ is a local orthonormal frame. This is nothing but $\mathcal{X}(\psi)$ for $X = \nabla f$. Hence we obtain the next proposition.

Proposition 2. *Let ψ be an L^2 eigenspinor for an eigenvalue λ. Then for every function f, every bounded domain D with smooth boundary in M, we have the equation*

$$\lambda \,\mathrm{Re} \int_D \langle \psi, \mathcal{H}_f(\psi) \rangle = -\lambda \,\mathrm{Re} \int_{\partial D} \langle \nu \cdot \psi, \nabla_{(\nabla f)} \psi \rangle.$$

If in addition M has a pole and $|\mathcal{H}_f|$ is bounded, then the following equation holds:

$$\lambda \,\mathrm{Re} \int_M \langle \psi, \mathcal{H}_f(\psi) \rangle = 0.$$

3. Non-existence of L^2 eigenspinors

Since the basic identity is found in the previous section, we have only to show the existence of appropriate vector fields or functions to prove the absence of eigenspinors. We perform this invoking some results in [4,6].

First, we consider a manifold with a pole p_0 which has a rotationally symmetric metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma(r)^2 \mathrm{d}\omega^2$$

with respect to the geodesic polar coordinate. In this equation, $\mathrm{d}\omega^2$ is the standard metric on the unit sphere in the Euclidean space. Because the metric is smooth, we have $\gamma(0) = 0$ and $\gamma'(0) = 1$.

From Proposition 5.5 in [4], the vector field $X = \gamma \frac{\partial}{\partial r}$ satisfies the identity

$$\nabla X = \gamma' \,\mathrm{Id}.$$

If ψ is an L^2 eigenspinor, then we have the equation

$$\mathcal{X}(\psi) = \gamma' \mathcal{D}\psi = \lambda \gamma' \psi$$

which implies

$$\lambda \,\mathrm{Re}\langle \psi, \mathcal{X}\psi \rangle = \lambda \gamma' |\psi|^2.$$

Because $\gamma' > 0$ near the point p_0, we get the following theorem.

Theorem 1. *Let M be a manifold with a pole which has a rotationally symmetric metric $\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma(r)^2 \mathrm{d}\omega^2$. If $\gamma' \geq 0$ and $\gamma \leq C_1 r + C_2$ for some positive constants C_1, C_2, then M has no L^2 eigenspinors for non-zero eigenvalues.*

Remark. Since $M \setminus \{p_0\}$ has a warped product metric, this (or even stronger) conclusion may be obtained by the method of the separation of variables.

Remark. The radial sectional curvature K_r of the rotationally symmetric metric is $K_r = -\frac{\gamma''}{\gamma}$, and $K_r \geq 0$ implies $\gamma' \geq 0$ ([8]).

Next, we take a bounded domain Ω in M and apply Proposition 2 to the exterior domain $M \setminus \Omega$. Lemma 2 can be proved for an eigenspinor ψ even if M is replaced by the exterior domain $M \setminus \Omega$ provided the spinor ψ satisfies the Dirichlet boundary condition on $\partial\Omega$. To be more precise, we consider the Dirac operator on $C_0^\infty(\mathrm{Int}(M \setminus \Omega))$ and its self-adjoint extension. Because

$$0 = \lambda \,\mathrm{Re} \int_{M \setminus \Omega} \langle \psi, \mathcal{X}(\psi) \rangle = \lambda \int_{M \setminus \Omega} \gamma' |\psi|^2,$$

we obtain the next theorem.

Theorem 2. *Let M be a manifolds with a pole which has a rotationally symmetric metric* $\mathrm{d}s^2 = \mathrm{d}r^2 + \gamma(r)^2\mathrm{d}\omega^2$, *and* Ω *be a bounded domain with smooth boundary in M. Assume that* $\gamma \le C_1 r + C_2$ *for positive constants* C_1, C_2 *on* $M \setminus \Omega$, *and that* (1) $\gamma' \le 0, \gamma' \not\equiv 0$ *on* $M \setminus \Omega$ *or* (2) $\gamma' \ge 0$, $\gamma' \not\equiv 0$ *on* $M \setminus \Omega$. *Then* $\mathcal{D}$ *with Dirichlet boundary condition on* $M \setminus \Omega$ *has no* L^2 *eigenspinors for non-zero eigenvalues.*

Now, we consider an n-dimensional complete manifold M which has a pole p_0 but is not necessarily rotationally symmetric. Let r and s denote the distance function from the point p_0 and the scalar curvature, respectively.

From the Schrödinger–Lichnerowicz formula

$$\mathcal{D}^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi,$$

we can deduce the following fact: *if the scalar curvature M is non-negative, then for any spinor field* $\psi \in W^1(\mathcal{S}) \cap C^\infty(\mathcal{S})$,

$$\int_M \|\mathcal{D}\psi\|^2 \ge \int_M \|\nabla\psi\|^2.$$

For a function f on M, let us take at each point an orthonormal frame $\{e_i\}$ so that the Hessian $\mathcal{H}_f$ is of diagonal form with respect to this frame. We denote

$$a_i = (\mathcal{H}_f)(e_i, e_i) \quad (1 \le i \le n)$$

with $a_1 = \min\{a_i\}$, $a_n = \max\{a_i\}$.

Lemma 3. *Assume that there exist positive constants* α *and* β *with*

$$\beta \le \mathcal{H}_f \le \alpha$$

everywhere on M, and that M has non-negative scalar curvature, then the following inequality holds for every spinor field ψ *in* $W^1(\mathcal{S}) \cap C^\infty(\mathcal{S})$:

$$\mathrm{Re}\int_M \langle \mathcal{D}\psi, \mathcal{H}_f(\psi)\rangle \ge \left[\left(1+\sqrt{n-1}\right)\beta - \sqrt{n-1}\alpha\right]\|\mathcal{D}\psi\|^2_{L^2(M)}.$$

If in addition ψ *is an eigenspinor of* $\mathcal{D}$ *for a real eigenvalue* λ, *then*

$$\lambda\,\mathrm{Re}\int_M \langle \psi, \mathcal{H}_f(\psi)\rangle \ge \left[\left(1+\sqrt{n-1}\right)\beta - \sqrt{n-1}\alpha\right]\lambda^2\|\psi\|^2_{L^2(M)}.$$

Proof. Let us define

$$A = \mathrm{Re}\sum_i a_i\langle \mathcal{D}\psi, e_i\cdot\nabla_{e_i}\psi\rangle - \mathrm{Re}\sum_i a_1\langle \mathcal{D}\psi, e_i\cdot\nabla_{e_i}\psi\rangle.$$

Then we obtain that

$$|A| \le \left|\sum_i (a_i - a_1)\langle \mathcal{D}\psi, e_i\cdot\nabla_{e_i}\psi\rangle\right| \le |\mathcal{D}\psi|\sqrt{\sum_i (a_i-a_1)^2}\sqrt{\sum_i |e_i\cdot\nabla_{e_i}\psi|^2}$$
$$\le \sqrt{n-1}(\alpha-\beta)|\mathcal{D}\psi||\nabla\psi|,$$

which implies the inequality

$$\int_M |A| \le \sqrt{n-1}(\alpha-\beta)\|\mathcal{D}\psi\|_{L^2}\|\nabla\psi\|_{L^2} \le \sqrt{n-1}(\alpha-\beta)\|\mathcal{D}\psi\|^2_{L^2}.$$

Hence, we have that

$$\mathrm{Re}\int_M \langle \mathcal{D}\psi, \mathcal{H}_f(\psi)\rangle = \mathrm{Re}\int_M \sum_i a_1 \langle \mathcal{D}\psi, e_i \cdot \nabla_{e_i}\psi\rangle + \int_M A \geq \beta \|\mathcal{D}\psi\|^2_{L^2(M)} - \int_M |A|$$
$$= \left[\left(1+\sqrt{n-1}\right)\beta - \sqrt{n-1}\alpha\right] \|\mathcal{D}\psi\|^2_{L^2(M)}.$$

Thus the proof is completed. □

Modifying the proof of Theorem 4.2 in [6], we can show the following theorem.

Theorem 3. *Let M be an n-dimensional Riemannian manifold with a pole p_0, and let r be the distance function from the point p_0. Assume that the radial sectional curvatures $K_r(x)$ of M at the point $x \in M$ satisfy*

$$0 \leq K_r(x) \leq \frac{c_n(1-c_n)}{r^2}$$

with

$$\frac{\sqrt{n-1}}{\sqrt{n-1}+1} < c_n < 1,$$

and that the scalar curvature of M is non-negative. Then the Dirac operator $\mathcal{D}$ has no L^2 eigenspinors for non-zero eigenvalues.

Proof. The function $f(x) = \frac{1}{2}r(x)^2$ is of class C^2 and $|\nabla f| \leq r$. Then we have

$$(\mathcal{H}_f)(X, X) = (X_r)^2 + r(\mathcal{H}_r)(X_T, X_T)$$

for every tangent vector X where

$$X = X_r \nabla r + X_T$$

is the orthogonal decomposition. Using Lemma 1.2 in [6], we get from the assumptions on the radial curvatures, the inequality

$$\frac{c_n}{r}|X_T|^2 \leq (\mathcal{H}_r)(X_T, X_T) \leq \frac{1}{r}|X_T|^2$$

which means in our terminology

$$a_n = 1, \qquad a_1 \geq c_n.$$

Hence the inequality

$$\left(1+\sqrt{n-1}\right)\beta - \sqrt{n-1}\,\alpha > 0$$

holds with $\beta = c_n$, $\alpha = 1$, and the proof is completed by Proposition 2 and Lemma 3. □

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